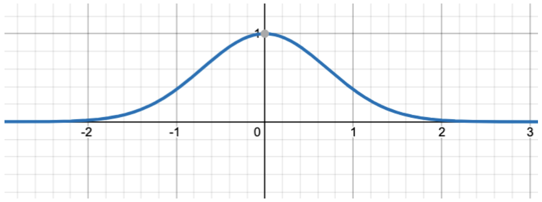
Chapter 11 Infinite Series



Compute $\int e^{-x^2}\,dx$

We will find that e^{-x^2} can be written as an "infinite ____________________"

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \frac{x^{10}}{120} + \cdots$$

$$\int e^{-x^2}\,dx = \int\left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \frac{x^{10}}{120} + \cdots\right)dx = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

So for example

$$\int_0^1 e^{-x^2}\,dx = \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \frac{x^{10}}{120} + \cdots\right)\Bigg|_0^1$$

So what is the BIG PICTURE?____

11.1-11.7 11.8-11.11

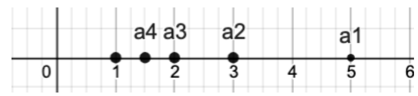
11.1 Sequences (light coverage)

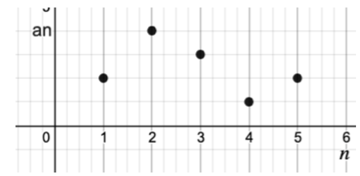
A sequence is a list of numbers in a definite order. $a_1, a_2, a_3, \ldots a_n, a_{n+1}, \ldots$

We can express sequences in many ways

$\{a_1, a_2, a_3, \ldots a_n, a_{n+1}, \ldots\}$ $\{a_n\}_{n=1}^{\infty}$ $a_n = f(n)$

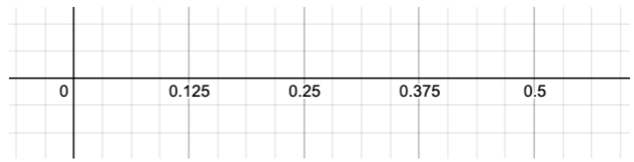
Domain?

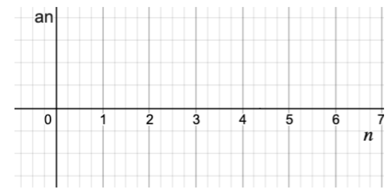




Example:

$\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \ldots\right\}$ __________. ____________





Examples: Find the general term, a_n , of the sequence if possible.

a) $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \ldots\right\}$

b) $\{1, -1, 1, -1 \ldots\}$

c) $\{1, 3, 5, 7 \ldots\}$

d) $\left\{\frac{1}{2}, \frac{5}{4}, \frac{7}{8}, \frac{17}{16}, \frac{31}{32} \ldots\right\}$

e) $\left\{\frac{\ln 1}{1}, \frac{\ln 2}{2}, \frac{\ln 3}{3}, \frac{\ln 4}{4} \ldots\right\}$

f) $\left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots\right\}$

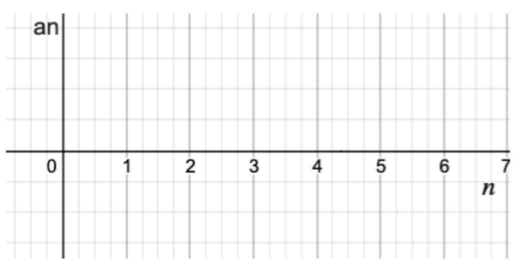
Convergence

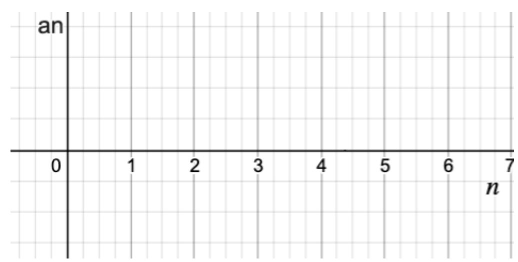
A sequence is said to ______________________ and have a _________________ L (finite) if for every $\varepsilon > 0$ there is an N>0 such that if n>N then ______________________________ If the sequence does not converge, it is said to ________________

Note: this is similar to $\lim\limits_{x \to} f(x) =$

How do we find limits of *sequences*? (consider example a)

Helpful Theorem: If $\lim\limits_{x \to \infty} f(x) = L$ (L finite) and $f(n) = a_n$ $\quad n = 1,2,3\ldots$ the $\lim\limits_{x \to \infty} a_n =$ ________ so the series converges.
Why? Is the converse true?





Determine whether the sequences in the above example converge

Example: Determine whether the sequence $\left\{ \frac{n!}{n^n} \right\}_{n=1}^{\infty}$ converges or diverges.

Sometimes we will need other tools to determine convergence of a sequence. Consider characteristics of certain sequences.

Monotonicity:

A sequence is ________________________________ if increasing or decreasing.

Example: Is $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ monotonic?

Many ways to determine. (Note: looking at what a finite number of terms do does not guarantee monotonic)

Example: Is $\left\{\frac{n^2}{n!}\right\}_{n=1}^{\infty}$ monotonic?

Boundedness

11 Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leqslant M \qquad \text{for all } n \geqslant 1$$

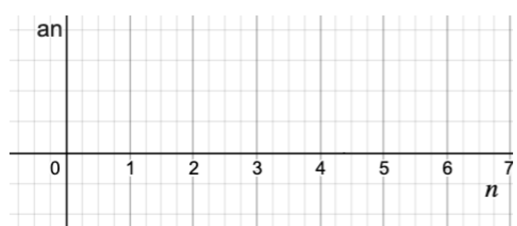
It is **bounded below** if there is a number m such that

$$m \leqslant a_n \qquad \text{for all } n \geqslant 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

Examples: Bounded?

$\{\sin(n)\}_{n=1}^{\infty}$ $\qquad$ $\{2n\}_{n=1}^{\infty}$ $\qquad$ $\{-2, 4, -6, 8, \ldots\}$ $\qquad$ $\left\{\frac{(-1)^n}{2^n}\right\}_{n=1}^{\infty}$



Suppose you have a monotonic, bounded sequence…

Theorem: Every bounded, monotonic sequence is convergent. (see proof in book or do proof for decreasing)

Use the above theorem to show that $\left\{\frac{5^n}{n!}\right\}_{n=1}^{\infty}$ is convergent.

Recursive Sequences:

Fibonnacci Sequence: $a_0 = 0, \quad a_1 = 1, \quad a_n = a_{n-1} + a_{n-2}$

11.2 Introduction to Infinite Series

Infinite series motivational example:

Stand 2 meters from the wall. At each turn step half the distance to the wall. Will you reach the wall? If so, what is the total distance traveled?

Turn # n	Length of step	Distance travelled =>	Running Total
1	1	1	
2	½	1+ 1/2	
3			7/4
4	1/8	1+ ½ + 1/6 + 1/8	15/8
5	1/16	1+ ½ + 1/6 + 1/8+ 1/16	31/36
:			
n		1+ ½ + 1/6 + 1/8+ 1/16…+ $1/2^{n-1}$	?????
:			

____.

So Total Distance (if we reach wall) would be $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$

See helpful tool on 5B page – sequence of partial sums desmos.

Definition: An infinite series is an expression of the form $a_1 + a_2 + a_3 + \cdots$ or $\sum_{n=1}^{\infty} a_n$. We wish to determine if the series has a "sum".

Let S_n be the sum of the first n terms ($S_1=a_1$, $S_2=a_1+a_2$ $S_3=a_1+a_2+a_3, \ldots$ $S_n=a_1+a_2+a_3+\ldots+a_n$). S_n is called the nth partial sum.
Now form the sequence of partial sums: S_1, S_2, $S_3, \ldots$. If this sequence of partial sum converges to a finite limit S (i.e.____________________) then the series is said to converge and the sum is defined to be S, that is $\sum_{n=1}^{\infty} a_n$ = S. Otherwise, the series diverges.

VOCABULARY/ NOTATION

General	Above Example
Series: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$	________________________________
Terms of the series: $a_1, a_2\ a_3, \cdots$	________________________________
nth term of the series: a_n	________________________________
nth partial sum: $s_n = a_1 + a_2 + a_3 + \cdots + a_n$	________________________________
sequence of partial sums: $s_1,\quad s_2,\quad s_3,$	________________________________
nth term partial sums: s_n in "closed form"	________________________________. ??

To determine convergence of series directly, using the definition:

1. Construct the ______________________________________

2. Find the general term of this sequence, ________________

   (This is usually the hard part. You need to find the pattern or perhaps the series is telescoping or you get otherwise creative.)

3. Check whether the sequence of partial sums converges by finding _________________.

If the limit is finite , say $\lim_{n\to\infty} s_n = S$, then the sequence of partial sums converges so the series converges and has sum S. $\left(\sum_{n=1}^{\infty} a_n = S\right)$. If the limit is infinite or does not exist, the sequence of partial sums diverges so the series diverges.

Example – two approaches. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$

1) Using the definition and forming a sequence of partial sums.

2) Telescoping Series

Example: $1 - 1 + 1 - 1 + 1 - 1 + 1\ldots$

Geometric Series

$$a + ar + ar^2 + ar^3 + \ldots = \sum_{n=1}^{\infty} \_\_\_\_\_\_\_\_$$

Examples: Given the following geometric series, write in sigma form and identify a and r
.

Converge? Sum?

1) $1 + \frac{1}{2} + \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \_\_\_\_\_ \quad a = \_\_ \quad r = \_\_$ ____________________

2) $3 + 6 + 12 + \cdots = \sum_{n=1}^{\infty} \_\_\_\_\_ \quad a = \_\_ \quad r = \_\_$ ____________________

3) $1 - 1 + 1 - \cdots = \sum_{n=1}^{\infty} \_\_\_\_\_ \quad a = \_\_ \quad r = \_\_$ ____________________

4) $1 + x + x^2 + x^3 + \cdots = \sum_{n=1}^{\infty} \_\_\_\_\_ \quad a = \_\_ \quad r = \_\_$ ____________________

Examples: Is the following a geometric series? If so, identify a and r

1) $\sum_{n=1}^{\infty} 4^n$

2) $\sum_{n=0}^{\infty} \frac{2}{5^{n+1}}$

3) $\sum_{n=1}^{\infty} 7n$

4) $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$

Convergence of Geometric Series

If $|r|$ =1: $\sum_{n=1}^{\infty} ar^{n-1} =$______________________________, s_n =____________ so series _______________

If $|r| \neq 1$ =1: s_n =__________________________________

rs_n =__________________________________

Subtracting, __________________________________

So s_n =__________________________________

Then $\lim_{n\to\infty} s_n$

4 The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \geqslant 1$, the geometric series is divergent.

Consider the convergence of the previous examples.

Example: Converge or diverge. $3+7+1+5+4+2+1+\frac{1}{2}+\frac{1}{4}+\cdots$

Convergence is not affected by a finite number of terms.

Interesting application: Can be used to find the exact value of repeating decimals.

$0.\overline{7}\overline{84}$ =

Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} =$$

This is a very important series that we will refer to often. Its name derives from the concept of overtones, or harmonics in music: the wavelengths of the overtones of a vibrating string are 1/2, 1/3, 1/4, etc., of the string's fundamental wavelength.

Does it converge? Is it geometric?

Back to basics…sequence of partial sums:

Does the desmos computer tool help?

We can prove, in fact, the harmonic series ________________________________

This proof, proposed by Nicole Oresme in around 1350, is considered by many in the mathematical community[by whom?] to be a high point of medieval mathematics. It is still a standard proof taught in mathematics classes today.

One way to prove divergence is to compare the harmonic series with another divergent series, where each denominator is replaced with the next-largest power of two:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots$$

$$\geq 1+\frac{1}{2}+\frac{1}{\mathbf{4}}+\frac{1}{4}+\frac{1}{\mathbf{8}}+\frac{1}{\mathbf{8}}+\frac{1}{\mathbf{8}}+\frac{1}{8}+\frac{1}{\mathbf{16}}+\cdots$$

$$1+\frac{1}{2}+\frac{1}{\mathbf{4}}+\frac{1}{4}+\frac{1}{\mathbf{8}}+\frac{1}{\mathbf{8}}+\frac{1}{\mathbf{8}}+\frac{1}{8}+\frac{1}{\mathbf{16}}+\cdots$$

More precisely,

$$S_{2^n} \geq 1+\frac{n}{2}$$

So the ________________________ diverges which means the _________________diverges.

The Test for Divergence (the nth term test)

You may have noticed that for a series to converge, the nth term must be getting small. In fact we can prove

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ is convergent the $\lim_{n\to\infty} a_n = 0$

Proof:

So,

$$\sum_{n=1}^{\infty} a_n \text{ is convergent } \Rightarrow \lim_{n\to\infty} a_n = 0$$

Caution, the converse is NOT TRUE,

$$\sum_{n=1}^{\infty} a_n \text{ is convergent } \qquad \lim_{n\to\infty} a_n = 0$$

Remember the harmonic series?

But, the contrapositive is true

$$\sum_{n=1}^{\infty} a_n \text{ is convergent } \Rightarrow \qquad \lim_{n\to\infty} a_n = 0$$

That is $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ is _______________

Exanples: Determine whether the following series converge.

$$\sum_{n=1}^{\infty} \frac{n}{n+1}$$

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$$

8 Theorem If $\Sigma\, a_n$ and $\Sigma\, b_n$ are convergent series, then so are the series $\Sigma\, ca_n$ (where c is a constant), $\Sigma\,(a_n + b_n)$, and $\Sigma\,(a_n - b_n)$, and

(i) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n(n+1)} + \frac{1}{2^{n-1}}\right)$$

Caution: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5n} \neq \frac{1}{5}\sum_{n=1}^{\infty} \frac{1}{n}$

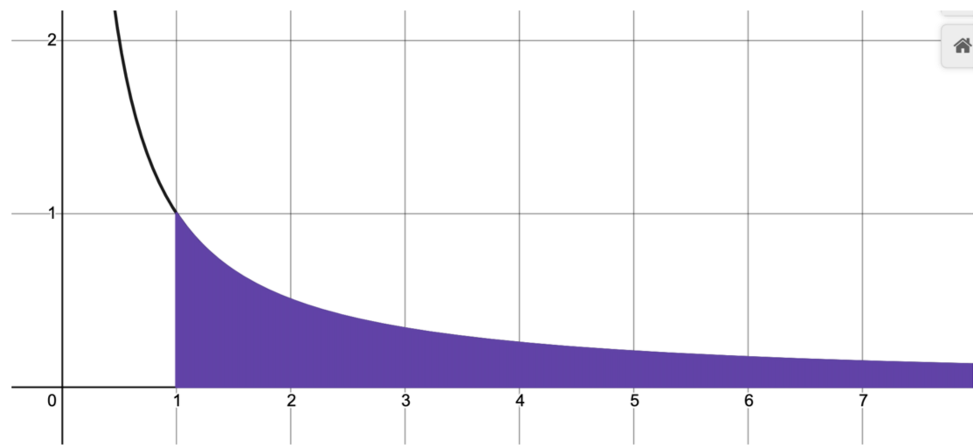
11.3 and 11.4: Convergence Tests for Positive (or ultimately positive) series.

If the series consists of positive terms, what can be said about the sequence of partial sums?

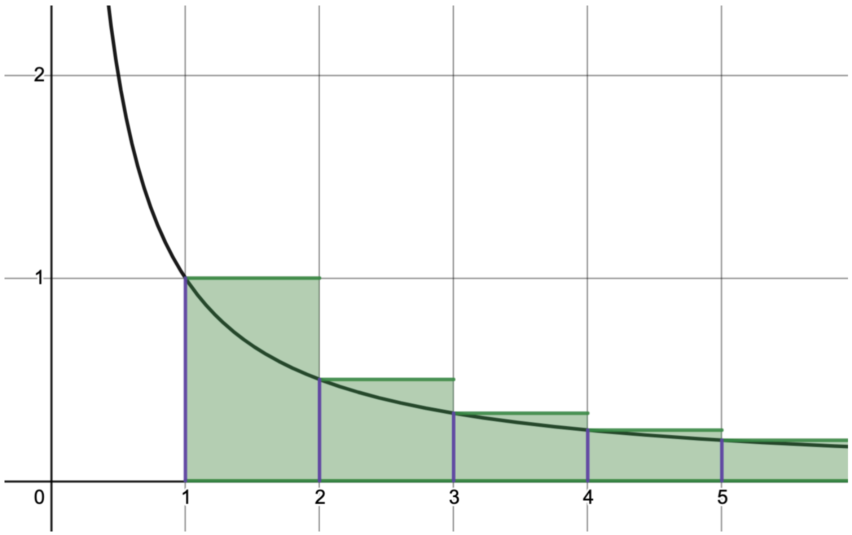
So for a positive series, we need only show the sequence of partial sums is ___________________________

Motivation Examples for the Integral Test.

(1) Consider $\sum_{n=1}^{\infty} \frac{1}{n}$ which we know ______________________________ and consider a seemingly unrelated problem: Find the area under $f(x) = \frac{1}{x}$ on $[1, \infty)$

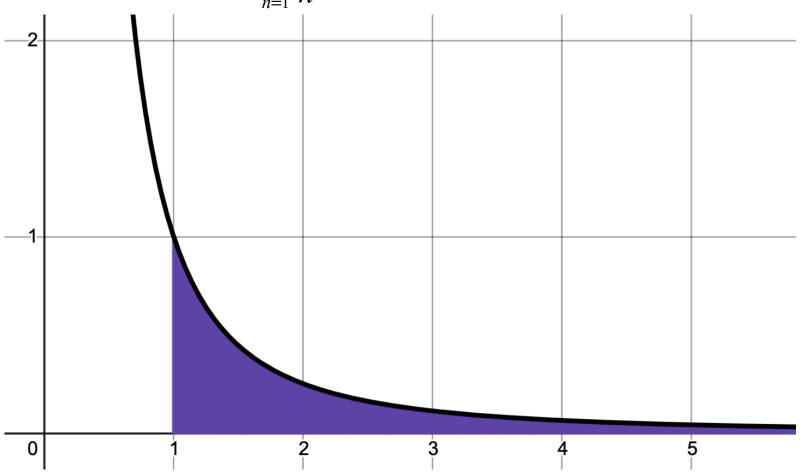


Suppose we have not discussed a way of computing this area so we instead approximate it using Riemann Sums with the sample point, x_i^* being the left endpoint of each subinterval and $\Delta x = 1$.



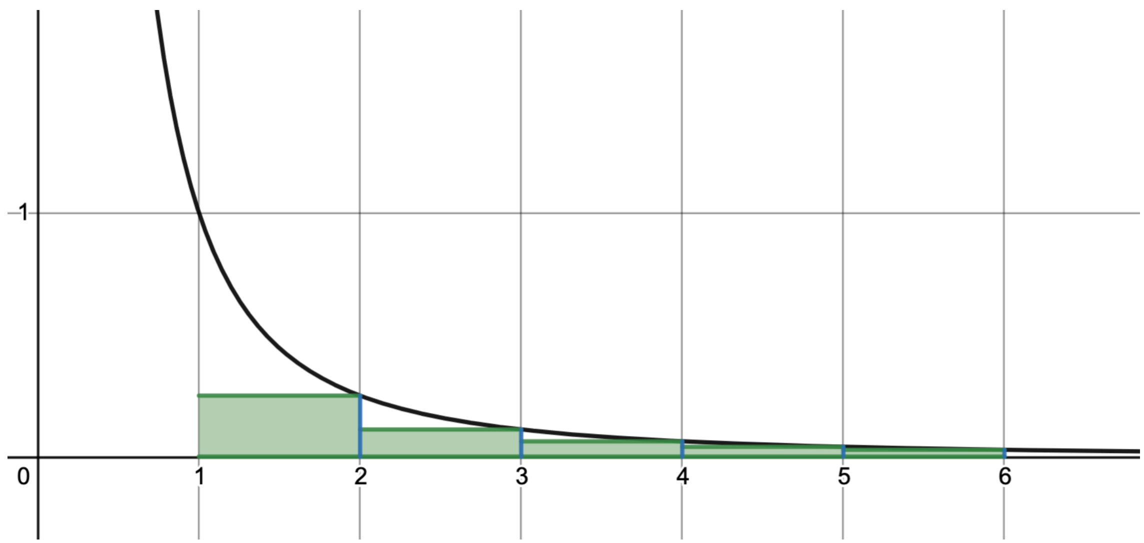
$$s_n > \int_1^{n+1} \frac{1}{x}\,dx$$

(2) Now consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and consider a seemingly unrelated problem: Find the area under $f(x) = \frac{1}{x^2}$ on $[1, \infty)$



We learned previously that $\int_1^{\infty} \frac{1}{x^2}\,dx = 1$. Suppose we wish to use this to somehow compare to the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Suppose we did not know the above fact and tried to approximate $\int_1^{\infty} \frac{1}{x^2}\,dx$ by again using Riemann Sums but this time we will use the sample point, x_i^* being the right endpoint of each subinterval and $\Delta x = 1$.



Note: The sum here is not 2, in fact it can be proved (at a higher level) to be ____________________.

These two examples suggest there is a relationship between the convergence of the ___________________ to the convergence of the similar

____

(Note: the choice of left/right sample point was made to illustrate the pre-known convergence. You won't be making this choice)

The Integral Test

Suppose $f(x)$ is a $\begin{cases} continuous \\ (ultimately)\, positive \\ (ultimately)\, decrea\sin g \end{cases}$ function on $[1,\infty)$ and let $a_n = f(n)$. Then

$$\sum_{n=1}^{\infty} a_n \quad converges \;. \Leftrightarrow \int_{1}^{\infty} f(x)\,dx \quad converges$$

Note:

- Integral and sum can start at N>1.
- We have not said integral and sum have same value. In fact, this does not tell us S.

Examples:

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

(2) $\sum_{n=1}^{\infty} ne^{-n^2}$

(3) $\sum_{n=1}^{\infty} \frac{1}{n^p}$

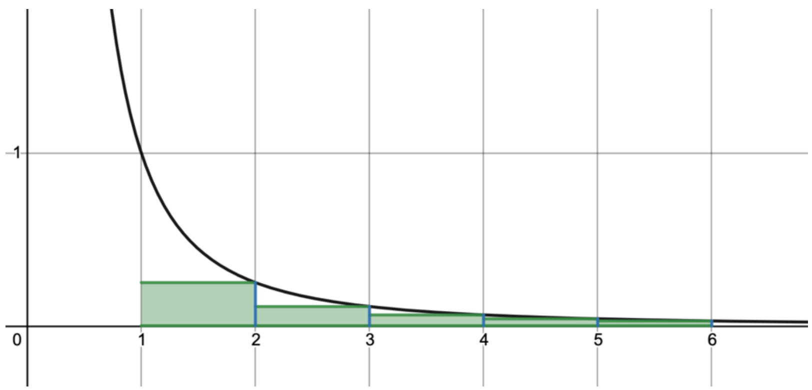
P-Series

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if p>1 and diverges otherwise.

Proof of the Integral Test

Consider $\sum_{n=1}^{\infty} a_n$ satisfying the 3 requirements of The Integral Test. If we approximate the integral $\int_1^{\infty} f(x)\,dx$ using:

Right Endpoints:



$$f(2)\Delta x + f(3)\Delta x + f(4)\Delta x + \cdots \_\_\_\_ \; Area$$

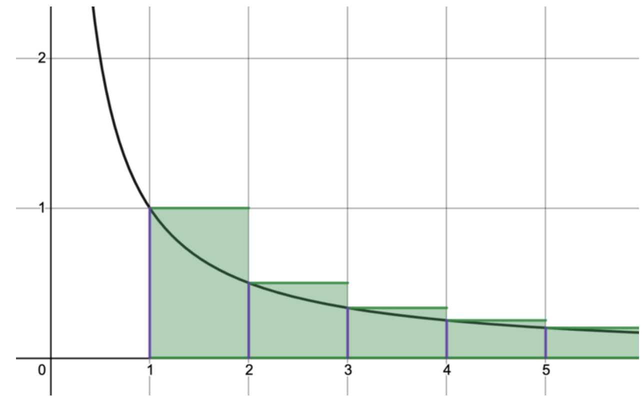
$$a_2 + a_3 + a_4 + \cdots < \_\_\_\_\_\_\_\_\_\_\_\_$$

$$\_\_\_\_\_\_ + a_2 + a_3 + a_4 + \cdots < \_\_\_\_\_\_\_\_\_\_\_\_$$

So if $\int_1^{\infty} f(x)\,dx$ converges, say $\int_1^{\infty} f(x)\,dx = L$

S_n

Left Endpoints



$$f(1)\Delta x + f(2)\Delta x + f(3)\Delta x + \cdots \_\_\_\_ \; Area$$

$$a_1 + a_2 + a_3 + \cdots > \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

if $\int_1^{\infty} f(x)\,dx$ diverges, in this case $\longrightarrow \infty$

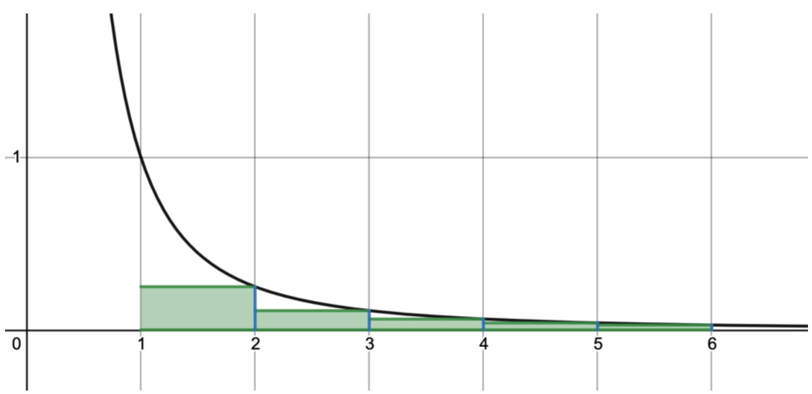
S_n

Estimating S and Error using the Integral Test

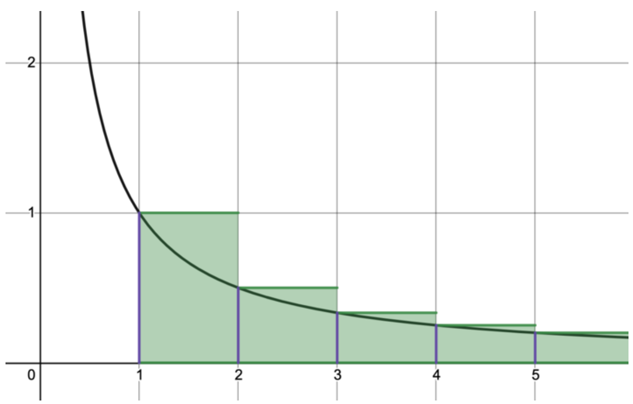
If $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + a_{n+1} + a_{n+2} + \cdots =$ ______________

So $R_n = a_{n+1} + a_{n+2} + \cdots$ is the error ____________________________________

What can we say about R_n ?



$R_n = a_{n+1} + a_{n+2} + \cdots <$ _________________



$R_n = a_{n+1} + a_{n+2} + \cdots >$ _______________

This leads to: $\int_{n+1}^{\infty} f(x)\,dx < R_n < \int_{n}^{\infty} f(x)\,dx$

This can be used in two ways: _

(1) __________________________ is used to find a bound on the error in using S_n to approximate S. It can also be used to find n for a specified error tolerance.

(2) The above inequality can be used to improve _________________________________ without using more terms.
Since $R_n = S - S_n$, the above inequality can be written:

$$\int_{n+1}^{\infty} f(x)\,dx < S - S_n < \int_{n}^{\infty} f(x)\,dx$$

Which leads to

$$\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ < S < \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Question: Suppose I am thinking of a number from 0 to 100. You try to guess, but for every point you are off, you have to pay $1.

Assuming you want to minimize the money you have to pay, what will you guess?_________________________

What is the most you will have to pay?______________________

Similarly, if A<S< B, the best guess for S is _______________________ and the most we would be off by is ___________________

That is, to improve our estimate of S without taking more terms, use:

$S \approx \dfrac{2S_n + \int_{n+1}^{\infty} f(x)\,dx + < \int_{n}^{\infty} f(x)\,dx}{2}$ which corresponds to a maximum error of $\dfrac{\int_{n}^{\infty} f(x)\,dx - < \int_{n+1}^{\infty} f(x)\,dx}{2}$

Example: Estimating S and Error using the Integral Test

Converge? $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

a) Estimate S by computing S_5

b) Estimate the error in using S_5 to approximate S

c) How many terms would be required so that the error is withing 0.01 when using S_n to approximate S

d) Use $S \approx \frac{2S_n + \int_{n+1}^{\infty} f(x)\,dx + < \int_{n}^{\infty} f(x)\,dx}{2}$ to improve the estimate of S in part a, without using more than 5 terms.

e) Estimate the error in part d. Error $\leq \frac{\int_{n}^{\infty} f(x)\,dx - < \int_{n+1}^{\infty} f(x)\,dx}{2}$

f) What would n have to be if we used the process in part (d) and desire error within 0.01